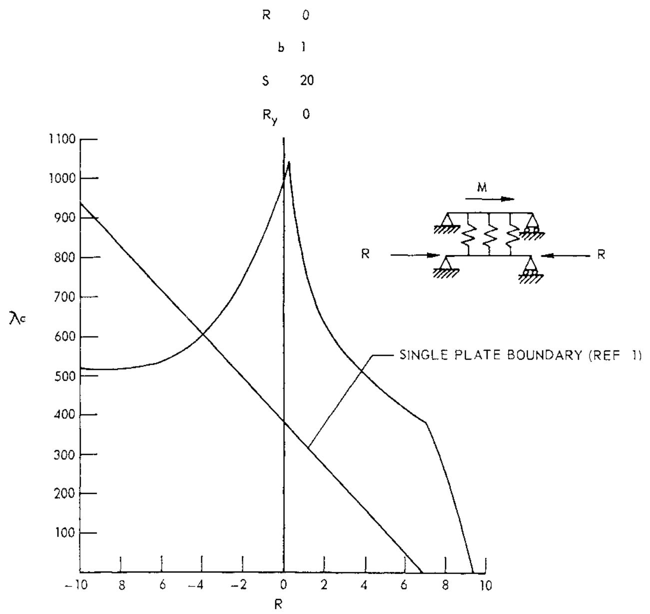

Fig 4 Flutter boundary for $R_{x+} = 0$

$= 1$, and $S_+ = S_- = S$ Flutter boundaries were derived from Eq (5) for several combinations of streamwise midplane loads in the two plates It must be remembered that these boundaries are subject to the same limitations that are described in Ref 1; in particular, the range of validity of the boundaries is limited by the buckling characteristics of the plates

Discussion

For a single flat isotropic plate, the flutter boundary as determined from a two-term Galerkin solution is a linear function of the midplane load in the streamwise direction Because of the coupling of the motions of the two plates, however, the configuration analyzed herein exhibits entirely different boundaries Peaks and valleys occur in the boundaries because the system can be tuned by means of the midplane loads

Figures 2–4 present flutter boundaries (λ_{cr} vs R_x) for square plates having various combinations of midplane loads and a spring parameter of $S = 20$ At the present time realistic values of S are not clearly defined; the value chosen ($S = 20$) might be typical of a configuration with a very soft filler material The flutter boundary for a single flat plate with the same physical properties as either the upper or the lower plate considered in the present analysis is also shown in each of the figures (see Ref 1)

Figure 2 is a plot of the boundary for the configuration when there is no load in the lower plate This boundary becomes asymptotic to the single plate boundary for large negative values of R_x, i e , large midplane tension, as do all of the boundaries considered herein

Figure 3 is a plot of the boundary when the midplane loads are the same in each plate Here the tuning effect can be very significant since it is possible to have a zero flutter speed with a tensile load, and a peak exists which is much higher than the corresponding value for the single plate

Figure 4 is a plot of the boundary when there is no load in the upper plate This case is perhaps the most realistic combination of loading if the configuration is considered to be a micrometeoroid protection device This boundary has characteristics very similar to the boundary in Fig 2 except that here the tuning effect is more prevalent In particular, for streamwise tension, a condition that can be expected from bending loads on a space vehicle during launch, the elastically supported plate is more prone to flutter than the single plate alone

The results of this analysis indicate that if a configuration similar to this one is used for applications where supersonic airflows are encountered, a very careful flutter analysis is in order to insure that undesirable flutter characteristics are not present

Reference

[1] Hedgepeth, J M , "Flutter of rectangular simply supported panels at high supersonic speeds," J Aeronaut Sci **24,** 563–573 (1957)

Re-Entry Heat Conduction of a Finite Slab with a Nonconstant Thermal Conductivity

William R Wells*
NASA Langley Research Center, Hampton, Va

The intense temperatures at the surface of a space vehicle may, in some instances, alter the thermal conductivity of the layers of the skin below the surface This analysis presents, as a first approximation to the problem, a closed-form solution for the case of a thermal conductivity that varies linearly with distance below the surface A convective heat input at the surface which is exponential in time is assumed The results should apply for the initial phases of entry in which the entry velocity and angle are nearly constant

Nomenclature

a = slope of thermal conductivity curve
B = constant that depends on entry velocity and angle
c = specific heat of material
C_1,C_2 = arbitrary constants
i = $-1^{1/2}$, imaginary unit
I_0,I_1 = modified Bessel functions of the first kind of order, zero and one, respectively
J_0,J_1 = Bessel functions of the first kind of order, zero and one, respectively
K = thermal conductivity
K_0,K_1 = modified Bessel functions of the second kind of order, zero and one, respectively
s = variable in the Laplace transform
t = time measured from entry
$T,\bar{T}$ = temperature and transformed temperature, respectively
V = entry velocity
x = distance normal to surface
Y_0,Y_1 = Bessel functions of the second kind of order, zero and one, respectively
γ = entry angle
θ = $\rho c/a^2$
ρ = density of material
τ = thickness of material
ω = constant that depends on entry velocity and angle

Subscripts

i = initial conditions
f = final conditions

The solution to the problem of one-dimensional heat conduction through a skin with constant thermal conductivity for the initial phases of a re-entry in which the velocity and entry angle are nearly constant and for which the convective heat rate is the dominant input can be found in Ref 1 The present analysis intends to take the basic

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* Aerospace Technologist, Space Mechanics Division

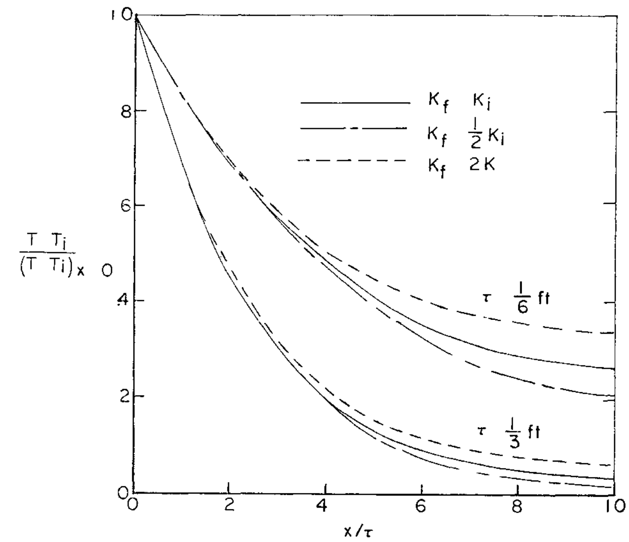

Fig 1 Temperatuie drop across skins having constant and nonconstant thermal conductivities

problem in Ref 1 and extend it to the case of a vehicle having a skin whose thermal conductivity varies linearly with depth below the surface Such a solution might serve as a first approximation to a situation in which the thermal conductivity of the layers of the skin below the surface are altered by the intense temperature at the surface

The temperature history through the skin can be obtained from the one-dimensional heat-conduction equation which is

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(K\frac{\partial T}{\partial x}\right) \tag{1}$$

The boundary conditions will be taken to be the same as those given in Ref 1, where it is assumed that, at the surface, the heat input is given by the convective heat rate that during the initial phase of entry can be represented by an exponential function in time The back side of the skin is insulated, and initially the skin is at a constant temperature throughout These conditions stated mathematically are

$$\left.\begin{aligned} \frac{\partial T}{\partial x}_{x=0} &= B(V,\gamma)\exp[\omega(V,\gamma)t] \\ \frac{\partial T}{\partial x}_{x=\tau} &= 0 \\ T(x,0) &= T_i \end{aligned}\right\} \tag{2}$$

$B(V,\gamma)$ and $\omega(V,\gamma)$ are constants that depend on the entry velocity and angle

We will assume that the thermal conductivity varies with depth as

$$K(x) = K_i + ax \tag{3}$$

where

$$a = (K_f - K_i)/\tau \tag{4}$$

K_i, K_f, and τ are the initial and final values for the thermal conductivity and skin thickness, respectively

Substitution of (3) into (1) gives

$$\rho c \frac{\partial T}{\partial t} = (K_i + ax)\frac{\partial^2 T}{\partial x^2} + a\frac{\partial T}{\partial x} \tag{5}$$

The solution of Eq (5) will be obtained by the application of the Laplace transform The transform of Eq (5) gives

$$K\frac{d^2\bar{T}}{dK^2} + \frac{d\bar{T}}{dK} - \theta s\bar{T} = -\theta T_i \tag{6}$$

where $\bar{T}(x,s)$ is the transform of $T(x,t)$ and $\theta \equiv \rho c/a^2$

The solution of (6) is

$$\bar{T}(x,s) = C_1 I_0[2(\theta sK)^{1/2}] + C_2 K_0[2(\theta sK)^{1/2}] + (T_i/s) \tag{7}$$

when I_0 and K_0 are the first and second kind of the modified Bessel functions of zero order

The transformations of the boundary conditions (2) are

$$\left.\begin{aligned} \frac{d\bar{T}}{dx}_{x=0} &= \frac{B}{s-\omega} \\ \frac{d\bar{T}}{dx}_{x=\tau} &= 0 \\ \bar{T}(x,0) &= T_i/s \end{aligned}\right\} \tag{8}$$

Use of Eq (8) in Eq (7) gives, for the transformed temperature,

$$\bar{T}(x,s) = \frac{T_i}{s} + \frac{B}{a(s-\omega)}\left(\frac{K_i}{\theta s}\right)^{1/2} \times \left\{\frac{K_1[2(\theta sK_f)^{1/2}]I_0[2(\theta sK)^{1/2}] + I_1[2(\theta sK_f)^{1/2}]K_0[2(\theta sK)^{1/2}]}{I_1[2(\theta_i K_i)^{1/2}]K_1[2(\theta sK_f)^{1/2}] - I_1[2(\theta sK_f)^{1/2}]K_1[2(\theta sK_i)^{1/2}]}\right\} \tag{9}$$

The poles of $\bar{T}(x,s)$ are at $s = 0$, ω, and the roots of the denominator of the expression in braces in Eq (9) These are all simple poles The apparent branch point corresponding to the factor $(K_i/\theta s)^{1/2}$ actually provides an extra contribution to the residue of the pole at $s = 0$, because of the behavior of the expression in braces as $s \to 0$

To find the poles associated with the expression in braces, we use the relations $I_0(x) = J_0(ix)$ and $K_0(x) = \frac{1}{2}\pi i[J_0(ix) + iY_0(ix)]$ These poles are then the roots of

$$J_1[2i(\theta sK_f)^{1/2}]Y_1[2i(\theta sK_i)^{1/2}] - J_1[2i(\theta sK_i)^{1/2}]Y_1[2i(\theta sK_f)^{1/2}] = 0 \tag{10}$$

J_1 and Y_1 are Bessel functions of the first and second kind of order one

Let β_m be the roots of (10) such that $is^{1/2} = \beta_n$, or

$$s = -\beta_m^2 \qquad (m = 1, 2, \quad) \tag{11}$$

The poles of $\bar{T}(x,s)$ are then at $s = 0$, ω, and $-\beta_m^2$ The values of β_m can be found in Ref 2 The temperature is then given as

$$T(x,t) = T_i + \frac{BK_i}{\rho c\omega\tau} + \frac{B}{a}\left(\frac{K_i}{\omega\theta}\right)^{1/2} \times e^{\omega t}\left\{\frac{K_1[2(\theta\omega K_f)^{1/2}]I_0[2(\theta\omega K)^{1/2}] + I_1[2(\theta\omega K_f)^{1/2}]K_0[2(\theta\omega K)^{1/2}]}{I_1[2(\theta\omega K_i)^{1/2}]K_1[2(\theta\omega K_f)^{1/2}] - I_1[2(\theta\omega K_f)^{1/2}]K_1[2(\theta\omega K_i)^{1/2}]}\right\}$$
$$+ \sum_{m=1}^{\infty} \frac{Be^{-\beta_m^2 t}}{a\theta(\omega + \beta_m^2)\Omega_m}\{J_0[2\beta_m(\theta K)^{1/2}]Y_1[2\beta_m(\theta K_f)^{1/2}] - Y_0[2\beta_m(\theta K)^{1/2}]J_1[2\beta_m(\theta K_f)^{1/2}]\} \tag{12}$$

where

$$\Omega_m = J_1[2\beta_m(\theta K_f)^{1/2}]Y_0[2\beta_m(\theta K_i)^{1/2}] - J_0[2\beta_m(\theta K_i)^{1/2}]Y_1[2\beta_m(\theta K_f)^{1/2}] + \left(\frac{K_f}{K_i}\right)^{1/2}\{Y_1[2\beta_m(\theta K_i)^{1/2}]J_0[2\beta_m(\theta K_f)^{1/2}] - J_1[2\beta_m(\theta K_i)^{1/2}]Y_0[2\beta_m(\theta K_f)^{1/2}]\} \tag{13}$$

Some numerical computations based on Eq (12) are shown in Figs 1 and 2 In Fig 1 a comparison is made of the stagnation-point temperature at any depth in the skin to that at the surface for the case of the thermal conductivity remaining constant, decreasing to one half its initial value, and

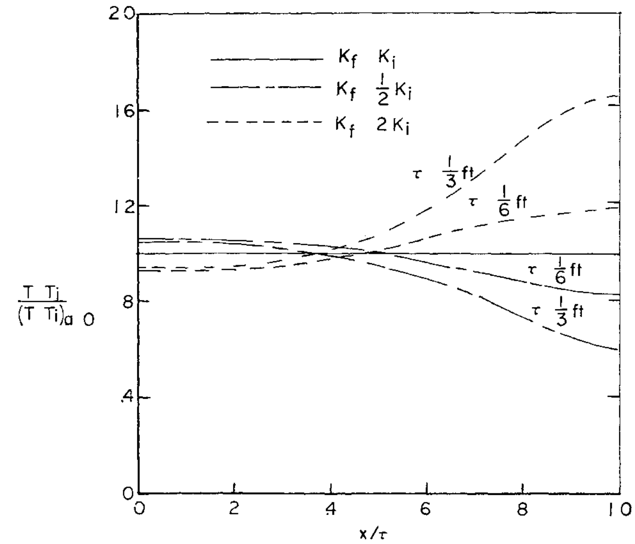

Fig 2 Comparison of temperatures in skins having constant and nonconstant thermal conductivities

increasing to twice its initial value Figure 2 shows a comparison of the temperature in a skin having a nonconstant thermal conductivity to one for which the thermal conductivity is constant The curves in Fig 2 show a slight relief in temperature up to about the middle of the skin for the case of the thermal conductivity increasing with depth However, for the back half of the skin, this variation in thermal conductivity shows a rapid increase in temperature On the other hand, there is a slight temperature rise in the first half of the skin for the case of decreasing thermal conductivity with an appreciable decrease in temperature for the back half of the skin for this variation in thermal conductivity

For these figures an entry velocity of 20,000 fps at 400,000 ft and an entry angle of $-20°$ was used The initial value of the thermal conductivity used was 0 548 Btu/ft-sec-°F which corresponds to electrolytic copper at 1000°F The results shown are time independent after about 10 sec

References

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[2] Jahnke, E and Emde, F, *Tables of Functions* (Dover Publications, Inc, New York, 1945), 4th ed, pp 204–206

Nth Order Solutions to Certain Thrust Integrals

Edward E Markson*
Martin Company, Baltimore, Md

An exact solution is presented to certain thrust integrals which previously have been solved with a linearization technique The solution is achieved by identifying two well known integral forms that are common to all components of the thrust integrals and whose series solutions converge for all values of the independent variable

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* Senior Engineering Specialist, Aerospace Mechanics Department Member AIAA

Nomenclature

A, B	= thrust integral defined specifically by Eq (2)
a	= constant involving summations of $\dot\theta$, $\dot\psi$, and ω
f	= function of sinat or cosat
F	= thrust vector
I	= integral function defined specifically by Eq (10)
J	= integral function defined specifically by Eq (9)
i, j, k	= mutually orthogonal inertial unit vectors
m	= vehicle mass
r	= central body radius vector to vehicle
t	= time
T	= vehicle burnup time, $(m_0/\dot m)$
x, y, z	= inertial Cartesian coordinate system
θ	= coangle between thrust vector and z axis
μ	= central body gravitational constant
ν	= value of v at time $= \tau$
τ	= burning time
v	= dummy variable
ψ	= angle between x axis and projection of **F** in x-y plane
ω	= mean motion $= \{\mu/[\frac{1}{2}(r_0 + r_t)]^3\}^{1/2}$

Subscripts

0	= at time zero
s	= function involving a sine
c	= function involving a cosine
i	= function involving the ith form of the constant a
x, y, z,	= in the direction of i j or k

Introduction

Reference 1 presents a solution to the vector differential equation of motion in a central force field:

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{\mu}{r^3}\mathbf{r} + \frac{\mathbf{F}}{m} \tag{1}$$

with the assumption that the change in (μ/r^3) is small with respect to the thrust acceleration vector This solution, though exact within the assumption as stated, requires the evaluation of certain thrust integrals **A** and **B** which have the following form:

$$\mathbf{A} = \frac{1}{\omega}\int_0^\tau \frac{\mathbf{F}}{m}\cos\omega t\, dt$$
$$\mathbf{B} = \frac{1}{\omega}\int_0^\tau \frac{\mathbf{F}}{m}\sin\omega t\, dt \tag{2}$$

Reference 2 develops a first order evaluation of **A** and **B** from the following formulation for **F**:

$$F_x(t) = F(t)\cos(\theta_0 + \dot\theta t)\cos(\psi_0 + \dot\psi t)$$
$$F_y(t) = F(t)\cos(\theta_0 + \dot\theta t)\sin(\psi_0 + \dot\psi t) \tag{3}$$
$$F_z(t) = F(t)\sin(\theta_0 + \dot\theta t)$$

and $\mathbf{F} = \mathbf{i}F_x + \mathbf{j}F_y - \mathbf{k}F$ where **i**, **j**, and **k** are mutually orthogonal inertial unit vectors Although the first-order solution[2] was adequate for the expressed intent, it is not necessary to restrict oneself to first order solutions for **A** and **B** To the contrary, nth order (i e, exact) solutions may be achieved and it is the purpose of this note to present these

Theory

Consider first the component of **A** in the $-\mathbf{k}$ direction:

$$-\omega A_z = \int_0^\tau \frac{F_z}{m}\cos\omega t\, dt \tag{4}$$

Also note that[1]

$$\omega\dot{\mathbf{A}} = (\mathbf{F}/m)\cos\omega t \qquad m = m_0 - \dot m t \qquad T = m_0/\dot m \tag{5}$$